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A characterization of double covers of curves in terms of the ample cone of second symmetric product

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ABSTRACT

We investigate the nef cone spanned by the diagonal and the fibre classes of second symmetric product of a curve of genus g. This 2-dimensional nef cone gives a characterization of double covers of curves of genus $\leq \frac{g-1}{8}$. This is a generalization of a result by Debarre [Olivier Debarre, Seshadri constants of abelian varieties, in: The Fano Conference, Univ. Torino, Turin, 2004, pp. 379–394, Proposition 8].

1. Introduction

Let C be a smooth integral curve of genus $g \geq 2$, $C^{(2)}$ the second symmetric product of C, and x and δ the fibre and the diagonal numerical classes respectively in the Néron–Severi space $\mathrm{N}^1(C^{(2)})_{\mathbf{R}}$ where the fibre class is the numerical class of the curve $C + a$ for any fixed point $a \in C$. We are interested in the restriction of the nef cone $\mathrm{Nef}(C^{(2)})$ on the plane spanned by x and δ. It has two boundaries, one is

$$(g-1)x + (\delta/2)$$

which is orthogonal to the diagonal class in the cone of effective curves on $C^{(2)}$, and the other is

$$(\tau(C)+1)x - (\delta/2)$$

where

$$\tau(C) := \inf\{\mu \geq 0 \mid (\mu+1)x - (\delta/2) \text{ is nef on } C^{(2)}\}.$$

Since the self-intersection of $(\tau(C)+1)x - (\delta/2)$ is non-negative,

$$\tau(C) \geq \sqrt{g}.$$

When C is very general, $\mathrm{N}^1(C^{(2)})_{\mathbf{R}}$ is generated by x and δ, and it was conjectured that $\tau(C) = \sqrt{g}$ for $g \geq 4$ and g is not a perfect square ([5, Section 3], [9, Conjecture 1.1]).

On the other hand, it is interesting to know what the value of $\tau(C)$ is when C is special. The first answer was given by Lazarsfeld [10, Example 1.5.13]. For $g \geq 3$,

$$\tau(C) = g \text{ if and only if } C \text{ is hyperelliptic.}$$

Among other things, the same result was also given by Debarre [6] and independently by Kong [7].

Actually, Debarre proved more in the same paper.

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Proposition 1.1 ([6, Proposition 8]). *Let C be a smooth projective curve of genus $g \geq 2$.*

(a) *If C hyperelliptic, then $\tau(C) = g$.*
(b) *If C is non-hyperelliptic, then:*
- *If $g = 3$, then $\tau(C) = \frac{9}{5}$.*
- *If $g = 4$, then $\tau(C) = 2$.*
- *If $g \geq 5$, then $\tau(C) \leq g - 2$ where the equality holds if and only if C is bielliptic.*

It characterizes the biellipticity of C with the value of $\tau(C) = g - 2$.
Along this direction, we give the main theorem in this paper.

Theorem 1.2. *Fix an integer $k \geq 0$. If C is a smooth integral curve of genus $g > \max\{2k+1, 4k-3\}$, then*

$\tau(C) \geq g - k$ *if and only if*
there exists a smooth integral curve H
of genus q with $q \leq \dfrac{k}{2}$ such that C is a double cover of H.

Furthermore in this case H is unique up to isomorphic and

$$\tau(C) = g - 2q.$$

For $g \geq 6$, choose the greatest integer k such that $\frac{g+3}{4} > k \geq 2$. Then $k \geq \frac{g-1}{4}$. The following is an immediate consequence.

Corollary 1.3. *Suppose C is a smooth irreducible curve of genus $g \geq 6$. We have*
(a) *If $\tau(C) \geq \frac{3g+1}{4}$, then*

$$C \text{ is a double cover of a curve of genus } \frac{g - \tau(C)}{2}.$$

(b) *If C is a double cover of curve of genus $q \leq \frac{g-1}{8}$, then*

$$\tau(C) = g - 2q.$$

More concisely, it gives a characterization of double covers of curves as the following.

Corollary 1.4. *Let $\mathcal{M}_g$ be the set of smooth integral curves of genus $g \geq 6$.*
Then,

$$\left\{ C \in \mathcal{M}_g | \tau(C) \geq \frac{3g+1}{4} \right\} = \left\{ C \in \mathcal{M}_g | C \text{ is a double cover of curve of genus } q \leq \frac{g-1}{8} \right\}.$$

In the last section, we compute the values of $\tau(C)$ for some other classes of special curves.

2. Curves induced by pencils

With the notations in the previous section, for computing the constant $\tau(C)$, we need to create some curves on the second symmetric products. We make use of the pencils for this purpose. In this section, we derive some basic properties about how these pencil curves are related to the value of $\tau(C)$.

On $\mathrm{N}^1(C^{(2)})_{\mathbf{R}}$, we have the following intersection equations

$$(x.x) = 1, \qquad (x.\delta) = 2, \qquad (\delta.\delta) = 4 - 4g. \tag{2.1}$$

For convenience of computation, for any divisor D on $C^{(2)}$, we can always find unique n and γ such that

$$D \equiv (n + \gamma)x - \gamma(\delta/2) + \xi$$

where $\xi \in < x, \delta >^{\perp}$. We set up the notation

$$(A, B) + \xi := (A + B)x - B(\delta/2) + \xi.$$

For divisors

$$D_1 \equiv (n_1, \gamma_1) + \xi_1 \quad \text{and} \quad D_2 \equiv (n_2, \gamma_2) + \xi_2,$$

by (2.1), we have

$$\begin{aligned} &(D_1.D_2) = n_1 n_2 - \gamma_1 \gamma_2 g + (\xi_1.\xi_2), \\ &D_1 \pm D_2 \equiv (n_1 \pm n_2, \gamma_1 \pm \gamma_2) + (\xi_1 \pm \xi_2). \end{aligned} \tag{2.2}$$

Definition 2.3. Suppose C admits a base-point-free pencil g^1_d of degree d. Then one can define a reduced curve on $C^{(2)}$ as

$$\Gamma(g^1_d) := \{a + b \in C^{(2)} \mid g^1_d - a - b \geq 0\}.$$

The numerical class of $\Gamma(g^1_d)$ in $N^1(C^{(2)})_{\mathbf{R}}$ is

$$d \cdot x - (\delta/2). \tag{2.4}$$

Lemma 2.5. *We have the following simple facts on the value of $\tau(C)$.*

(a) $\tau(C) \geq \sqrt{g}$.
(b) *For any curve $C^{(2)} \supset \gamma \equiv (n, \gamma) + \xi$, n must be a positive integer.*
(c) *Assume C admits a base-point-free pencil g^1_d of degree d. Then,*
- $\tau(C) \geq \frac{g}{d-1}$.
- *If $d < 1 + \sqrt{g}$, then $\tau(C) > \sqrt{g}$.*
- *If $d \geq 1 + \sqrt{g}$ and the curve $\Gamma(g^1_d)$ is irreducible, then $\tau(C) \leq d - 1$.*

(d) *For any smooth curve C, $\tau(C) \leq g$.*

Proof. (a) Since $(\tau(C), 1)$ is nef, $(\tau(C), 1).(\tau(C), 1) = \tau(C)^2 - g \geq 0$.
(b) For any curve $C^{(2)} \supset \gamma \equiv (n, \gamma) + \xi$, since x is ample, $(x.C) = n$ which must be a positive integer.
- From (2.4) we know $\Gamma(g^1_d) \equiv (d - 1, 1)$ and
  $(\tau(C), 1).(d - 1, 1) = \tau(C)(d - 1) - g \geq 0$
  by the definition of $\tau(C)$.
- Trivial consequence from the last result.
- Since $\Gamma(g^1_d)^2 = (d - 1, 1)^2 = (d - 1)^2 - g \geq 0$, then $\Gamma(g^1_d) \equiv (d - 1, 1)$ is nef, and hence $\tau(C) \leq d - 1$.

(c) It is well-known that any C always admits a base-point-free pencil A of degree $g + 1$ and the $\Gamma(A)$ is irreducible ([3] Lemma 1.3 and Lemma 1.4 (i)), then $\tau(C) \leq g$ from the above. □

In [8], Kouvidakis proved the following.

Proposition 2.6. *Suppose C is a smooth integral projective curve and C admits a base-point-free pencil g^1_d of degree d. If $d \leq 1 + [\sqrt{g}]$ and $\Gamma(g^1_d)$ is irreducible, then*

$$\tau(C) = \frac{g}{d - 1}.$$

Kouvidakis's result means that if a smooth integral curve C has a low degree base-point-free pencil and the curve induced by the pencil on $C^{(2)}$ is irreducible, then the class of $\Gamma(g^1_d)$ is orthogonal to the boundary $(\tau(C) + 1)x - (\delta/2)$ and it computes $\tau(C)$.

Actually, this fact is also true even when the induced curve is reducible. To this end, we have to consider the situation of an arbitrary effective divisor on $C^{(2)}$.

First, we need the following modified version of a lemma by Ross.

Lemma 2.7 ([9, Lemma 2.2]). *Suppose C is a smooth integral curve of genus $g \geq 2$. We have,*

(a) *If $\tau(C) > \sqrt{g}$, then there exists an irreducible curve Γ on $C^{(2)}$ such that $\Gamma \equiv (n, \gamma) + \xi$ with $\xi \in <x, \delta>^{\perp}$, $\gamma\sqrt{g} > n$ and $\tau(C) = \frac{\gamma g}{n}$ (i.e. this curve Γ computes $\tau(C)$).*
(b) *If there exists an effective divisor D on $C^{(2)}$ such that $D \equiv (a, b)$ with integers $a, b > 0$, then either $\tau(C) \leq a/b$, or an irreducible component of* Supp(D) *computes $\tau(C)$. In the latter situation,*

$$\tau(C) = \max_{C' \subset \mathrm{Supp}(D)} \{\mu \mid \mu = R(C')\}$$

where the maximum is taken over all irreducible components of Supp(D) *and*

$$R(C') := \frac{(\delta.C') - 2(x.C')}{2(x.C')}.$$

In particular, if C admits a base-point-free pencil g^1_e and $\tau(C) > e - 1$, then

$$\tau(C) = \max_{C' \subset \Gamma(g^1_e)} \{R(C')\}$$

where the maximum is taken over all irreducible components of $\Gamma(g^1_e)$.

Proof. For (a) Since $(\tau(C)+1)x-(\delta/2)$ is nef and not ample and since $(\tau(C),1)^2>0$, by the Nakai criterion for real divisors, there must exist an irreducible curve Γ on $C^{(2)}$ such that

$$((\tau(C)+1)x-(\delta/2)).\Gamma=0.$$

Suppose $\Gamma\equiv(n,\gamma)+\xi$. Then the result follows.

For (b) Assume $\tau(C)>a/b$. Since $D\equiv(a,b)$ is effective,

$$\tau(C)a-bg=(\tau(C),1).(a,b)\geq 0.$$

Thus, $\tau(C)\geq\frac{bg}{a}>\frac{g}{\tau(C)}$ and hence $\tau(C)>\sqrt{g}$. From (a) an irreducible curve Γ computing $\tau(C)$ exists and

$$D.\Gamma=an-b\gamma g=an-bn\tau(C)<0.$$

Hence, Γ is a component of $\mathrm{Supp}(D)$.

Suppose $C'\equiv(n',\gamma')+\xi'$ is one of the components. Since C' is an integral curve, we have

$$\tau(C)n'-\gamma'g=(\tau(C),1).((n',\gamma')+\xi')\geq 0\Rightarrow\tau(C)\geq\frac{\gamma'g}{n'}. \tag{2.8}$$

Since $(x.C')=n'$ and $(\delta.C')=2n'+2\gamma'g$, then

$$\frac{\gamma'g}{n'}=\frac{(\delta.C')-2(x.C')}{2(x.C')}.$$

Hence, $\tau(C)=\max_{C'\subset\mathrm{Supp}(D)}\{R(C')\}$.

For the particular statement, since $\Gamma(g^1_e)\equiv(e-1,1)$, as above $\tau(C)>\sqrt{g}$ and it must be computed by one of the irreducible components of $\Gamma(g^1_e)$ and

$$\tau(C)=\max_{C'\subset\Gamma(g^1_e)}\{R(C')\}.\quad\square$$

Corollary 2.9. *Suppose C is a smooth integral curve of genus $g\geq 2$. If there exists an effective divisor D on $C^{(2)}$ such that $D\equiv(a,b)$ with a,b being positive integers and $D^2\leq 0$, then either $\tau(C)=\sqrt{g}=\frac{a}{b}$, or an irreducible component of* $\mathrm{Supp}(D)$ *computes $\tau(C)$.*

Proof. It is clear that $D^2\leq 0$ implies $a\leq b\sqrt{g}$.

For the case $\tau(C)>\sqrt{g}\geq a/b$, Lemma 2.7(b) implies an irreducible component of $\mathrm{Supp}(D)$ computing $\tau(C)$.

For $\tau(C)=\sqrt{g}$, since $a\tau(C)\geq bg$ (by the effectiveness of D), $a=b\sqrt{g}$ and g is a perfect square. We get

$$\tau(C)=\sqrt{g}=\frac{a}{b}.\quad\square \tag{2.10}$$

Remark 2.11. The result of Corollary 2.9 implies Proposition 2.6. To this end, since $\Gamma(g^1_d)\equiv(d-1,1)$, then Corollary 2.9 implies either

$$\tau(C)=\sqrt{g},\ d-1=\sqrt{g}\quad\text{and } g \text{ is a perfect square},$$

or one of the components of $\Gamma(g^1_d)$ computes $\tau(C)$. Since $\Gamma(g^1_d)$ is irreducible, it itself computes

$$\tau(C)=g/(d-1).$$

3. Covering curves

In this section, we give the sufficient conditions that a curve on the nth symmetric product of C is irreducible and smooth, and takes C as a covering of degree n. Let us recall some definitions and equations from [2] we will use in the following.

Let C be a smooth irreducible curve of genus g, $J(C)$ its Jacobian variety and $C^{(d)}$ its d-fold symmetric product. From the classical theory, fixing a point $p\in C$, we can define the morphisms

$$\begin{aligned} &u_d:C^{(d)}\to J(C)\quad\text{where } u_d(D)=\mathcal{O}(D-d\cdot p)\text{ and } D\in C^{(d)},\\ &i_{d-1}:C^{(d-1)}\to C^{(d)}\quad\text{where } i_{d-1}(D)=D+p\text{ and } D\in C^{(d-1)}.\end{aligned} \tag{3.1}$$

We denote the theta divisor on $J(C)$ by Θ and its numerical class in the Néron–Severi space by θ. On $C^{(d)}$ we denote the class of $u_d^*(\theta)$ by θ_d or simply θ if there is no confusion in the context and the class of $i_{d-1}(C^{(d-1)})$ in $C^{(d)}$ by x_d or simply x.

Suppose $\sigma:C^{(n-1)}\times C\to C^{(n)}$ is the natural surjective morphism.

Set $D(n,p)=\sigma(C^{(n-1)},p)$. Clearly,

$$D(n,p)=i_{n-1}(C^{(n-1)})\equiv x\quad\text{for any } p\in C. \tag{3.2}$$

Consider the morphism

$$\phi : C^{(d-2)} \times C \to C^{(d)}$$

defined by $\phi(D, p) = D + 2p$. The image of ϕ is the *diagonal* δ_d in $C^{(d)}$. Then [2, p. 358, Proposition 5.1] states that

$$\delta_d \equiv 2((d+g-1)x - \theta). \tag{3.3}$$

Given an algebraic cycle Z in $C^{(d)}$ we define two associated cycles

$$\begin{aligned} A_k(Z) &= \{E \in C^{(d+k)} : E - D \geq 0 \text{ for some } D \in Z\}, \\ B_k(Z) &= \{E \in C^{(d-k)} : D - E \geq 0 \text{ for some } D \in Z\}. \end{aligned} \tag{3.4}$$

With all the above, we compute the intersection numbers for the curves from coverings.

Suppose $\pi : C \to H$ is an n-sheeted covering of C over a smooth irreducible curve H of genus h. Then we have a curve $\Sigma := \{\pi^{-1}(q) : q \in H\} \subset C^{(n)}$. We have the following intersection numbers.

Lemma 3.5 (*[2, p. 370 D-10]*).

$$\begin{aligned} (x.\Sigma) &= 1, \\ (\delta.\Sigma) &= 2(g-1) - 2n(h-1) \\ &= \textit{the degree of the ramification divisor of } \pi, \\ (\theta.\Sigma) &= nh. \end{aligned}$$

Conversely, under some hypotheses, C can be an n-sheeted covering. The following lemma is from [4, 1.5]. We modified the conditions appropriately for our purpose.

Lemma 3.6. *For any integral curve $\gamma \subset C^{(n)}$, if it satisfies*

(i) $\gamma \not\subset \mathrm{Supp}(D(n, p))$ *for all* $p \in C$ *and*
(ii) $(x.\gamma) = 1$,

then γ is smooth and there is a degree n covering $f : C \to \gamma$. In particular, $\Sigma := \{f^{-1}(q) : q \in \gamma\} \subset C^{(n)}$ is smooth and isomorphic to γ.

Proof. Fix $p \in C$. Suppose $\eta \in \gamma \cap D(n, p)$. Since $\gamma \not\subset \mathrm{Supp}D(n, p)$,

$$1 = (x.\gamma) = D(n, p).\gamma \geq \mathrm{mult}_\eta D(p) \cdot \mathrm{mult}_\eta \gamma = \mathrm{mult}_\eta \gamma.$$

This is true any $p \in C$. Therefore γ intersects $D(n, p)$ only at η and γ is smooth at η.

Set-theoretically, a map $f : C \to \gamma$ can be defined by

$$f(p) = D(n, p) \cap \gamma.$$

Consider the natural morphism $\sigma : C^{(n-1)} \times C \to C^{(n)}$ and let $p_2 : C^{(n-1)} \times C \to C$ denote the second factor projection.

Since $D(n, p) = \sigma(C^{(n-1)} \times p)$ and σ is a finite morphism, then the inverse image $\sigma^{-1}(\gamma)$ of γ is also a curve. By the projection formula of intersection of cycles,

$$\sigma^{-1}(\gamma).(C^{(n-1)} \times p) = \gamma.D(p) = 1.$$

Therefore there is only one integral component of $\sigma^{-1}(\gamma)$ dominant C through p_2, and this morphism is finite and of degree one. Since C is smooth, that integral component is isomorphic to C.

On the other hand, the hypothesis $\gamma \not\subset \mathrm{Supp}(D(n, p))$ for all p guarantees that no component of $\sigma^{-1}(\gamma)$ can be contained in any fibre of p_2. Thus, $\sigma^{-1}(\gamma)$ is irreducible and isomorphic to C through p_2.

Then, we can define f as,

$$f := \sigma \circ (p_2|_{\sigma^{-1}(\gamma)})^{-1}.$$

Since σ is of degree n, f is a degree n covering.

By the definition of Σ, we have a natural birational morphism from γ into Σ. Since Σ comes from a degree n covering, Σ cannot be contained in the support of $D(n, p)$ for any $p \in C$. By Lemma 3.5 and the above arguments, Σ is smooth and hence isomorphic to γ. □

In particular, on $C^{(2)}$, those integral curves, not a curve $C + a$ for any fixed a, of intersection number one with the fibre class x have C as a double cover.

Remark 3.7. With the same result of Lemma 3.6, the hypotheses in [4, 1.5] are:

(a) $\gamma \not\subset \delta_n$.
(b) For any $p \in C$, $D(n, p)$ cuts γ in one single point.

Actually, our conditions imply these two above. To this end, assume to the contrary of (a) that $\gamma \subset \delta_n$. A general point on γ is $D + 2p_0$ where $p_0 \in C$ and D is an effective divisor on C of degree $n-2$. Then, there exists an integral curve α in $C^{(n-1)} \times C$ with general point on it being $(D+p_0, p_0)$ and $\sigma(\alpha) = \gamma$. The morphism σ is ramified along α. Since $\gamma \not\subset \mathrm{Supp}(D(n,p))$ for all $p \in C$, then α is dominant C through p_2. Thus, we have

$$(\gamma.x) = \gamma.D(n,p) = \sigma^{-1}(\gamma).(C^{(n-1)} \times p) \geq 2\alpha.(C^{(n-1)} \times p) \geq 2.$$

Contradiction.

For (b), it can be proved similarly as in the proof of Lemma 3.6.

Again suppose $\pi : C \to H$ is an n-sheeted covering of C over a smooth irreducible curve H of genus h and we define $\Sigma := \{\pi^{-1}(q) : q \in H\} \subset C^{(n)}$. We would like to apply the two operations (3.4) to the curve Σ. By Lemma 3.6, $H \cong \Sigma$. Thus, H is embedded into $C^{(n)}$. From the definitions, $B_{n-d}(H) = B_{n-d}(\Sigma)$ is a 1-cycle in $C^{(d)}$ for any $2 \leq d \leq n$. Moreover, with Lemma 3.5 and [2, p. 368 D-2 and D-8], ones have the following intersection numbers.

Lemma 3.8.

$$(x.B_{n-d}(H)) = \binom{n-1}{n-d},$$

$$(\theta.B_{n-d}(H)) = nh\binom{n-2}{n-d} + g\binom{n-2}{n-d-1},$$

$$(\delta.B_{n-d}(H)) = 2(d+g-1)\binom{n-1}{n-d} - 2\left(nh\binom{n-2}{n-d} + g\binom{n-2}{n-d-1}\right).$$

In particular, for $d = 2$,

$$\begin{aligned}(x.B_{n-2}(H)) &= n-1,\\ (\delta.B_{n-2}(H)) &= 2(g-1) - 2n(h-1) \\ &= \text{the degree of the ramification divisor of } \pi.\end{aligned} \tag{3.9}$$

Corollary 3.10. *Under the above settings, if Γ is an irreducible component of $B_{n-d}(H)$ for some $2 \leq d \leq n$ and $(x.\Gamma) = 1$, then C is a d-sheeted cover of Γ and π factorizes through this cover.*

Proof. Since $C \to H$ is a covering, $\Gamma \not\subset \mathrm{Supp}(D(d,p))$ for all $p \in C$. From Lemma 3.6, $C \to \Gamma$ is a covering of degree d. By the definition of $B_{n-d}(H)$, $\pi : C \to H$ must factorize through Γ. □

4. Characterization of double cover curves

The Lemma 3.6 tells the geometric meaning of intersection number one with the fibre class x. This leads us a way to characterize those double covering curves. To give the proof of Theorem 1.2, we need two more well-known results.

Theorem 4.1 (*[1, Theorem 3.5]*). *Let C, C_1 and C_2 be smooth projective integral curves of genera g, g_1 and g_2 respectively. Let $\pi_1 : C \to C_1$ and $\pi_2 : C \to C_2$ be n_1- and n_2-sheeted coverings. Assume there does not exist a smooth integral curve Γ of genus $h < g$ with coverings $f : C \to \Gamma$ and $\alpha_i : \Gamma \to C_i$ such that*

$$\pi_i = \alpha_i \circ f \quad \text{for } i = 1, 2.$$

Then,

$$g \leq n_1 g_1 + n_2 g_2 + (n_1 - 1)(n_2 - 1).$$

Let $W_d^r(C)$ denote the subvariety of $\mathrm{Pic}^d(C)$ parameterizing complete linear series of degree d and dimension at least r:

$$W_d^r(C) = \{|D| : \deg D = d \text{ and } r(C) \geq r\}.$$

Lemma 4.2 (*[2, p. 181, 3.3]*). *Suppose $r \geq d - g$. Then every component of $W_d^r(C)$ has dimension greater than or equal to the Brill–Noether number*

$$\rho(g, r, d) = g - (r+1)(g - d + r).$$

The following gives the core part of Theorem 1.2.

Proposition 4.3. *Fix an integer $k \geq 0$. Suppose C is a smooth integral curve of genus $g > 2k+1$. Assume C admits a base-point-free pencil g_d^1 of degree $d \geq 2$ with $d < g - 2k + 3$. Furthermore, assume $\Gamma(g_d^1)$ contains an irreducible component computing $\tau(C)$. Then,*

$$\tau(C) \geq g - k \quad \textit{if and only if}$$
$$C \textit{ is a double cover of a smooth integral curve } H$$
$$\textit{of genus } q \textit{ with } q \leq \frac{k}{2}.$$

Furthermore in this case H is isomorphic to the component of $\Gamma(g_d^1)$ computing $\tau(C)$ and

$$\tau(C) = g - 2q.$$

Proof. Suppose Σ is the irreducible component of $\Gamma(g_d^1)$ computing $\tau(C)$. Then,

$$\tau(C) = \frac{(\delta.\Sigma) - 2(x.\Sigma)}{2(x.\Sigma)}.$$

For any irreducible component C' of $\Gamma(g_d^1)$, let us set

$$R(C') := \frac{(\delta.C') - 2(x.C')}{2(x.C')}.$$

It means that $\tau(C) = R(\Sigma)$.

It is clear that $(x.\gamma) \geq 1$ for any integral curve γ on $C^{(2)}$.

It splits into two cases.

(i) For $(x.C') = 1$. Since $C' \subset \Gamma(g_d^1)$ and g_d^1 is base-point-free, $C' \not\subset D(2, p)$ for any fixed $p \in C$. By Lemma 3.6, this induces a double covering of C onto C'. By Lemma 3.5,

$$(\delta.C') = 2g - 4g' + 2$$

where g' is the genus of C'. This computes

$$R(C') = g - 2g'.$$

(ii) $(x.C') \geq 2$. Since $\Gamma(g_d^1)$ comes from covering of C, there is no component of $\Gamma(g_d^1)$ that can be the diagonal of $C^{(2)}$. Thus, $2d - 2 + 2g = (\delta.\Gamma(g_d^1)) \geq (\delta.C') \geq 0$. Since $d < g - 2k + 3$, this gives

$$R(C') \leq \frac{(d-1) + g - 2}{2} < g - k.$$

"$\Leftarrow$": Suppose $C \to H$ is a double cover of a curve H of genus q for $0 \leq q \leq \frac{k}{2}$. Then, H can be embedded into $C^{(2)}$ and $((\tau(C) + 1)x - (\delta/2)).H \geq 0$. Thus, from Lemma 3.5,

$$\tau(C) \geq g - 2q \geq g - k.$$

The irreducible component Σ must satisfy (i) and hence

$$\tau(C) = g - 2g' \geq g - 2q \quad \text{and} \quad 0 \leq g' \leq q \leq \frac{k}{2}.$$

If H and C' are not isomorphic, then $C \to C'$ and $C \to H$ are two distinct double covers, by Theorem 4.1,

$$g \leq 2q + 2g' + 1.$$

However, $2q + 2g' + 1 \leq 2k + 1 < g$. Contradiction. Hence, $H \cong C'$ and

$$\tau(C) = R(C') = g - 2g' = g - 2q.$$

"$\Rightarrow$": Assume $\tau(C) \geq g - k$. Again the component Σ must be in case (i). Then,

$$\tau(C) = R(C') = g - 2g'.$$

Thus, $0 \leq g' \leq \frac{k}{2}$. □

If g is large enough, there always exists a base-point-free pencil satisfying the hypotheses of Proposition 4.3. This is a direct application of the Brill–Noether number.

Proof of Theorem 1.2. Let d be the gonality of C, i.e. d is the smallest integer such that C has a base-point-free degree d pencil. By Lemma 4.2,

$$2 \leq d \leq \left[\frac{g+3}{2}\right].$$

Let g^1_d denote a base-point-free pencil of degree d. Since $g > 4k-3$ and $g > 2k+1$, we have the following inequalities respectively,

$$d \le \frac{g+3}{2} < g - 2k + 3$$

and

$$d - 1 \le \frac{g+3}{2} - 1 < g - k.$$

To apply Proposition 4.3, we have to check $\Gamma(g^1_d)$ contains a component computing $\tau(C)$.

"$\Rightarrow$": Assume $\tau(C) \ge g-k$, then

$$\tau(C) > d - 1.$$

Thus Lemma 2.7 implies that a component of $\Gamma(g^1_d)$ computes $\tau(C)$. Then, Proposition 4.3 implies the conclusion.

"$\Leftarrow$": If $C \to H$ is a double cover of a curve H of genus q for $q \le \frac{k}{2}$, then H can be embedded into $C^{(2)}$ and $((\tau(C)+1)x - (\delta/2)).H \ge 0$. Thus,

$$\tau(C) \ge g - 2q \ge g - k.$$

Again, Lemma 2.7 implies a component of $\Gamma(g^1_d)$ to compute $\tau(C)$. Then the conclusion follows from Proposition 4.3. □

Remark 4.4. (i) The bound $\max\{4k-3, 2k+1\} = 4k-3$ when $k \ge 2$. We put it in the theorem for the completeness to include the hyperelliptic cases of $2 \le g \le 5$.

(ii) For the bielliptic case $k = 2$, Theorem 1.2(b) needs $g > 5$ to conclude C is a bielliptic curve from $\tau(C) = g-2$. However, in Proposition 1.1, Debarre proved that if $g = 5$ and $\tau(C) = g-2 = 3$, then C is bielliptic. This is a boundary situation where we cannot use the inequality

$$d < g - 2k + 3$$

to eliminate case (ii) in the proof of Proposition 4.3 when $g = 5$, $k = 2$ and $d = 4$. Instead, Debarre used an analysis on the orbits of the Galois group G of the base-point-free g^1_4 on C. By this way the irreducible component of $\Gamma(g^1_4)$ computing $\tau(C)$ must have intersection number one with the fibre class x, i.e. must satisfy case (i) in the proof of Proposition 4.3. Then going through the arguments followed in the proof, one has the conclusion that C is bielliptic when $g = 5$ and $\tau(C) = 3$.

5. Examples

If an integral curve C is a cover of higher degree, the value of $\tau(C)$ can jump down to equal or less than half its genus. We are computing some examples to illustrate this situation.

Suppose C is a smooth integral curve of genus g. For any integral curve γ on $C^{(2)}$, keep using the notation

$$R(\gamma) := \frac{(\delta.\gamma) - 2(x.\gamma)}{2(x.\gamma)}.$$

(I) Suppose C is a trigonal of genus $g \ge 4$ (i.e. non-hyperelliptic and admits a base-point-free pencil g^1_3). From (3.9),

$$(x.\Gamma(g^1_3)) = 2.$$

If $\Gamma(g^1_3)$ is reducible, it has at most two components. Let C' be one of them, then $(x.C') = 1$. However, Corollary 3.10 implies the morphism by g^1_3 factorizing through C'. This is impossible as 3 is prime. Hence, $\Gamma(g^1_3)$ is irreducible. For $g \ge 4$, by Proposition 2.6,

$$\tau(C) = \frac{g}{2}.$$

(II) [6, Remark 9] Suppose C is a tetragonal of genus $g \ge 9$. If $\tau(C) = \sqrt{g}$, then (2.10) implies

$$\frac{g}{d-1} = \tau(C).$$

Then, $g = 9$ and $\tau(C) = 3$.

Assume $\tau(C) > \sqrt{g}$.

If $\Gamma(g^1_4)$ is irreducible, then by Proposition 2.6

$$\tau(C) = \frac{g}{3}.$$

If it is reducible, then Lemma 2.7 implies one of the components computing $\tau(C)$. Since

$$(x.\Gamma(g_4^1)) = 1 + 2 = 1 + 1 + 1,$$

$\Gamma(g_4^1)$ can contain two or three components and in each case the morphism by g_4^1 factorizes through a 2-sheeted covering onto a curve H by Corollary 3.10. We can use Lemma 3.5 and Lemma 2.7 to compute $\tau(C)$.

For the case "$1 + 2$", H computes $g - 2h$ and $\Gamma(g_4^1) - H$ computes h, thus

$$\tau(C) = \max\{h, g - 2h\}$$

where h is the genus of H.

For the case "$1 + 1 + 1$",

$$\tau(C) = \max\{g - 2h_1, g - 2h_2, g - 2h_3\}$$

where h_i is the genus of H_i and H_i's are the components of $\Gamma(g_4^1)$ for $i = 1, 2, 3$.

(III) Suppose C is a 5-gonal of genus $g \geq 16$. Since 5 is prime, $\Gamma(g_5^1)$ cannot decompose into a component C' with $(x.C') = 1$. Since $(x.\Gamma(g_5^1)) = 4$, $\Gamma(g_5^1)$ can only decompose into at most two components and they have intersection number with x being 2. Let C' be one of them, then

$$\frac{(\delta.C') - 2(x.C')}{2(x.C')} \leq \frac{(\delta.\Gamma(g_5^1)) - 2(x.C')}{2(x.C')} \leq \frac{g+2}{2}.$$

Thus,

$$\frac{g}{4} \leq \tau(C) \leq \frac{g+2}{2}.$$

(IV) Suppose C is a triple cover of a curve H of genus h and H admits a base-point-free pencil h_2^1 (i.e. elliptic or hyperelliptic). The composition of the triple cover and the h_2^1 on H is a base-point-free pencil g_6^1 on C. We would like to find a lower bound l of g such that

$$\tau(C) = \frac{g - 3h}{2}$$

when $g \geq l$. First $g > (6-1)^2 = 25$ such that $\tau(C) > \sqrt{g}$. Then, by Lemma 2.7 an irreducible component of $\Gamma(g_6^1)$ computes $\tau(C)$. Since g_6^1 is compounded by H,

$$\Gamma(g_6^1) \supset B_1(H)$$

(the definition of $B_k(Z)$ in the above). From (3.9), $(x.B_1(H)) = 2$. Using the similar argument as for trigonals, we can see that $B_1(H)$ is irreducible. Then,

$$R(B_1(H)) = \frac{g - 3h}{2}.$$

For the other components in $\Gamma(g_6^1)$, there are three cases.

$x.(\Gamma(g_6^1) - B_1(H)) = 3 = 1 + 2 = 1 + 1 + 1$.

For the case "3" and $\Gamma(g_6^1) - B_1(H) = \Gamma$,

$$R(\Gamma) = h.$$

For the case "$1 + 2$" and $\Gamma(g_6^1) - B_1(H) = \Gamma_1 + \Gamma_2$, we have

$$R(\Gamma_1) = g - 2g_1 \quad \text{and} \quad R(\Gamma_2) = \frac{3h - g + 2g_1}{2}$$

where $C \to \Gamma_1$ is a double cover and g_1 is the genus of Γ_1. For the case "$1 + 1 + 1$" and $\Gamma(g_6^1) - B_1(H) = \Gamma_1 + \Gamma_2 + \Gamma_3$, we have

$$R(\Gamma_i) = g - 2g_i$$

where $C \to \Gamma_i$ is a double cover and g_i is the genus of Γ_i for $i = 1, 2, 3$.

By Theorem 4.1, we have

$$g - 2g_i \leq 2 + 3h$$

for those double cover components in all cases.

By Hurwitz's formula,

$$g + 1 \geq 2g_i.$$

This implies in the case "$1 + 2$",

$$R(\Gamma_2) \leq \frac{3h + 1}{2}.$$

Thus, to make

$$R(B_1(H)) \geq \max\left\{h, 2+3h, \frac{3h+1}{2}\right\}$$

it requires $g \geq 9h+4$. Hence, $l = \max\{9h+4, 26\}$.

In particular, if C is a triple cover of an elliptic curve, then

$$\tau(C) = \frac{g-3}{2} \quad \text{if } g \geq 26.$$

The critical point to prove Proposition 4.3 is Lemma 3.6, which tells us very well that when $(x.\Gamma) = 1$ we have a double cover from C onto Γ. However, we cannot control those components with $(x.\Gamma) \geq 2$. Although we know they might come from a higher degree covering of C onto some curve Lemma 3.8, the intersection number is an necessary condition but not sufficient. For instance, in the example (III) above, it is possible that $B_3(\Gamma(g_5^1))$ has a component Γ with $(x.\Gamma) = 2$ but g_5^1 cannot be factorized.

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